J. Fluid Mech. (1997), *vol.* 353, *pp.* 197–203. Printed in the United Kingdom

Stratification-induced lateral dispersion of a density anomaly

By **RONALD SMITH**

Department of Mathematical Sciences, Loughborough University, LE11 3TU, UK

(Received 24 March 1997 and in revised form 9 July 1997)

When a small amount of marked solute is released into a stratified fluid, there is lateral dispersion of the marked solute and a larger lateral dispersion of any density anomaly. The method of moments is used to calculate the two dispersion coefficients. The excess dispersion for the density is shown to be proportional to the fractional density decrease from the bed to the free surface and to the cube of the water depth, and inversely proportional to the vertical mixing for lateral momentum. For weak turbulent mixing the stratification-induced lateral dispersion for the density anomaly can be several orders of magnitude greater than the lateral turbulent mixing for marked solute

1. Introduction

In a lecture, W. R. Geyer pointed out that in the Hudson estuary off New York, the salinity disturbance from a hot fresh outfall is observed to spread very much more rapidly than the associated temperature disturbance. This totally disagreed with model calculations that I had given (Smith 1978*a*,*b*) which predicted that in well-mixed estuaries the dispersion processes for perturbations to the salinity or temperature are the same because the two constituents experience the same turbulent mixing in the same buoyancy-perturbed flow. Since the Hudson is moderately stratified (Nepf & Geyer 1996; Geyer & Nepf 1997), there was a reason for the major disagreement between Geyer's observations and my calculations. The purpose of this short paper is to make that reason quantitative by calculating the stratification-induced difference D^* between the lateral dispersion coefficients for different constituents.

The physical basis of the calculations is a distinction between the dispersion of marked particles and the dispersion of a density anomaly. Vertical displacements in a stratified fluid allow density adjustments remote from any discharged solute. For the Hudson estuary, the stratification is principally associated with salinity. Thus, a localized heat input would be dispersed at the rate for marked particles. If the associated salinity input were isotopically distinct from the background stratification, then the isotopically marked salinity would likewise spread at the rate for marked particles. However, if the salinity input is indistinguishable from the background salinity, then the salinity disturbance would appear to disperse at the larger rate associated with the density adjustment. For the Hudson the difference between the two dispersion rates is found to be extremely large.

2. Perturbation equations

It is assumed that, prior to any discharge, there is water with constant depth H and stratified density $\rho_0(1+\rho'(z))$. The Boussinesq approximation will be made, in which the large magnitude of the gravitational acceleration g allows changes in weight $g\rho_0\rho'(z)$ to be dynamically significant even though the density varies by only a very small fraction $\rho'(z)$ relative to the the reference density ρ_0. With the sign convention of the z-coordinate increasing upwards, gravitational stability requires that $\partial_z\rho' \leqslant 0$. Perturbations to the process whereby the background stratification is maintained will be ignored. Also, it will be assumed that the perturbations have large length-to-width ratio, so that in a suitable moving frame of reference the longitudinal x-structure can be ignored. A quasi-laminar turbulence model will be used with eddy viscosities ν_2, ν_3 and eddy diffusivities κ_2, κ_3 in the transverse and vertical directions.

The perturbation quantities associated with a small discharge of marked solute will be indicated by lower-case variables. The perturbation velocity $(0, v(y,z,t), w(y,z,t))$, pressure $\rho_0 p(y,z,t)$, marked particle concentration $c(y,z,t)$ and density $\rho_0\delta(y,z,t)$ satisfy the water volume, transverse momentum, vertical momentum, diffusion and density perturbation equations:

$$\partial_y v + \partial_z w = \partial_t q \ , \tag{2.1}$$

$$\partial_t v + \partial_y p - \partial_y(\nu_2\partial_y v) - \partial_z(\nu_3\partial_z v) = \partial_t m_2 \ , \tag{2.2}$$

$$\partial_t w + g\delta + \partial_z p - \partial_y(\nu_2\partial_y w) - \partial_z(\nu_3\partial_z w) = \partial_t m_3 \ , \tag{2.3}$$

$$\partial_t c - \partial_y(\kappa_2\partial_y c) - \partial_z(\kappa_3\partial_z c) = \partial_t \gamma \ , \tag{2.4}$$

$$\partial_t\delta + w\partial_z\rho' = \alpha\partial_t c - \rho'\partial_t q \ . \tag{2.5}$$

The source terms represent transient fluxes of volume $\partial_t q(y,z,t)$, lateral momentum $\partial_t m_2(y,z,t)$, vertical momentum $\partial_t m_3(y,z,t)$, marked solute $\partial_t\gamma(y,z,t)$ and a linear relationship between concentration and density excess. The sign of the density coefficient α in equation (2.5) depends upon whether the marked solute increases or decreases the density (salt or temperature). Perturbations to the eddy viscosities and diffusivities (together with turbulence modelling equations for those perturbations) have been ignored.

The discharge can cause a small vertical displacement $\zeta(y,t)$ of the free surface. At the displaced free surface there is no vertical separation between that surface and the water, zero tangential stress, zero normal stress and zero loss of marked solute. For small-amplitude disturbances, these boundary conditions can be projected to the undisturbed surface position:

$$\left.\begin{array}{l} w - \partial_t\zeta = 0 \\ \nu_3(\partial_z v + \partial_y w) = 0 \\ -p + g\zeta + 2\nu_3\partial_z w = 0 \\ \kappa_3\partial_z c = 0 \end{array}\right\} \text{on } z = H \ . \tag{2.6}$$

At the flat, horizontal bed there is zero vertical flow, no slip, and zero normal flux of marked solute:

$$\left.\begin{array}{l} w = 0 \\ v = 0 \\ \kappa_3\partial_z c = 0 \end{array}\right\} \text{on } z = 0 \ . \tag{2.7}$$

Equations (2.1)–(2.7) are the equations and boundary conditions for small-amplitude internal or surface waves with the inclusion of viscosity, diffusion and source terms.

3. Lateral moments

Despite linearity and the numerous other simplifications, the strongly coupled system of equations (2.1)–(2.7) does not permit explicit solution. Following the now classical shear dispersion calculations of Aris (1956), any fine details of the y-dependence are sacrificed by our confining attention to the first few lateral moments:

$$c^{(n)}(z,t) = \int_{-\infty}^{\infty} c(y,z,t)y^n \mathrm{d}y, \quad n = 0,\ 1,\ 2,\ \ldots. \tag{3.1}$$

The moments of the volume, momentum, concentration and density equations (2.1)–(2.5) do not involve the y-coordinate but do have additional right-hand-side source-like terms involving the lower-order $^{(n-1)}$ or $^{(n-2)}$ moments:

$$\partial_z w^{(n)} = \partial_t q^{(n)} + n v^{(n-1)} , \tag{3.2}$$

$$\partial_t v^{(n)} - \partial_z(\nu_3 \partial_z v^{(n)}) = \partial_t m_2^{(n)} + n p^{(n-1)} + n(n-1)\nu_2 v^{(n-2)} , \tag{3.3}$$

$$\partial_t w^{(n)} - \partial_z(\nu_3 \partial_z w^{(n)}) + g\delta^{(n)} + \partial_z p^{(n)} = \partial_t m_3^{(n)} + n(n-1)\nu_2 w^{(n-2)} , \tag{3.4}$$

$$\partial_t c^{(n)} - \partial_z(\kappa_3 \partial_z c^{(n)}) = \partial_t \gamma^{(n)} + n(n-1)\kappa_2 c^{(n-2)} , \tag{3.5}$$

$$\partial_t \delta^{(n)} + w^{(n)} \partial_z \rho' = \alpha \partial_t c^{(n)} - \rho' \partial_t q^{(n)} . \tag{3.6}$$

It deserves comment that a right-hand-side lower-order term such as $n(n-1)\nu_2 v^{(n-2)}$ is absent until $n \geqslant 2$. The moments of the boundary conditions (2.6)–(2.7) merely require $^{(n)}$ superscripts, with the exception of the free-surface tangential stress:

$$\nu_2 \partial_z v^{(n)} = n\nu_2 w^{(n-1)} \text{ on } z = H . \tag{3.7}$$

It will be assumed that the discharge is transient and of small lateral extent centred at $y = 0$. Consequently, the transient flux terms can be ignored, except for the time-integrated jolts given to the $n = 0$ moments of volume

$$q^{(0)}(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_t q(y,z,t) \mathrm{d}t \, \mathrm{d}y , \tag{3.8}$$

lateral momentum $m_2^{(0)}(z)$, vertical momentum $m_3^{(0)}(z)$ and concentration $\gamma^{(0)}(z)$.

4. Long-time solutions

The next simplification is to remove the time coordinate t. From the $n = 0$ version of equation (3.2) and the corresponding kinematic free-surface condition, we can reproduce the conservation of volume result that the mean displacement of the free surface exactly accommodates the total volume of water from the discharge:

$$\zeta^{(0)} \sim \int_0^H q^{(0)}(z) \mathrm{d}z = H\bar{q}^{(0)} . \tag{4.1}$$

An overbar is used to indicate the average value over the water depth.

On a time scale long compared with those for the transient discharging, viscous decay of internal waves or vertical mixing of the marked solute, the zero moments of the perturbation lateral and vertical velocity will have dissipated to zero, the zero moment of free-surface displacement will have become constant, the zero moment of the perturbation pressure will have become hydrostatic, and the zero moments of the marked concentration and the density perturbation will have become vertically uniform :

$$\left.\begin{aligned} v^{(0)} \sim 0\,, \quad w^{(0)} \sim 0\,, \quad p^{(0)} \sim gH\bar{q}^{(0)} + g(H-z)[\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)}]\,, \\ c^{(0)} \sim \bar{\gamma}^{(0)}\,, \quad \delta^{(0)} \sim [\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)}]\,. \end{aligned}\right\} \tag{4.2}$$

The asymptote $v^{(0)} \sim 0$ corresponds to antisymmetry about $y = 0$ for the laterally outwards movement of the water in accommodating the discharge.

The long-time solutions for the zero moments provide the right-hand-side forcing terms in the equations for the $n = 1$ first moments. The assumed small lateral extent makes the first moments asymptotically zero with the exception of the pressure-driven lateral velocity $v^{(1)}$, which we decompose into volume and buoyancy contributions:

$$v^{(1)} \sim \; H\left\{\bar{q}^{(0)}V(z) + [\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)}][B(z) + \xi V(z)]\right\}\,, \tag{4.3}$$

with

$$V(z) = g\int_0^z \frac{H-z'}{v_3}\mathrm{d}z' \geqslant 0, \tag{4.4}$$

and

$$B(z) = g\int_0^z \left[\frac{H-z'}{2H} - \xi\right]\frac{H-z'}{v_3}\mathrm{d}z'. \tag{4.5}$$

At moderately large distances from the centreline $y = 0$, there is an outflow with velocity profile $V(z)$ to accommodate the excess volume of fluid and a buoyancy-driven contribution with velocity profile $B(z)$. The positive coefficient ξ is chosen so that there is zero vertically integrated transverse volume flux associated with the buoyancy-driven velocity profile $B(z)$:

$$\int_0^H B(z')\mathrm{d}z' = 0, \quad \text{i.e.} \quad \xi H\int_0^H \frac{(H-z)^2}{v_3}\mathrm{d}z = \int_0^H \frac{(H-z)^3}{2v_3}\mathrm{d}z\,. \tag{4.6}$$

The specifications (4.5), (4.6) imply that throughout the water column, the incomplete integral from the bed for the buoyancy-driven flow $B(z)$ is non-negative:

$$\int_0^z B(z')\mathrm{d}z' \geqslant 0\,. \tag{4.7}$$

It follows from equation (4.5) that as z increases from zero at the bed, so does $B(z)$. There is outwards buoyancy-driven flow which reaches its maximum not far above the bed at $z = H(1-2\xi)$. Upwards from that level $B(z)$ decreases and changes sign in the upper half of the water column to become an exactly compensating inflow near the surface.

The long-time solutions for the zero and first moments provide the right-hand-side forcing terms for the $n = 2$ second moments. From equation (3.2) it follows that the second moment of the vertical velocity has the large-time asymptote

$$w^{(2)} \sim \; 2H\left\{\bar{q}^{(0)} + \xi(\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)})\right\}\int_0^z V(z')\mathrm{d}z' + 2H(\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)})\int_0^z B(z')\mathrm{d}z'. \tag{4.8}$$

At large distances from the centreline $y = 0$, there is upwards movement associated with the volume of the discharge and with any density excess. The velocity at the free surface gives the rate of increase $\partial_t \zeta^{(2)}$ for the second moment of the free-surface displacement:

$$\partial_t \zeta^{(2)} \sim 2H \left\{ \bar{q}^{(0)} + \xi(\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)}) \right\} \int_0^H V(z')\mathrm{d}z'. \tag{4.9}$$

At large times, the vertical average of the second-moment equation (3.5) for concentration is

$$\partial_t \bar{c}^{(2)} \sim \; 2\bar{\kappa}_2 \bar{c}^{(0)} \; . \tag{4.10}$$

Thus, the lateral variance of the marked concentration grows diffusively with an effective lateral diffusivity equal to the vertical average lateral eddy diffusivity $\bar{\kappa}_2$.

At large times, the vertical average of the second-moment equation (3.6) for the density perturbation is

$$\partial_t \bar{\delta}^{(2)} \sim 2g \left\{ \bar{q}^{(0)} + \xi(\alpha\bar{\gamma}^{(0)} - \overline{\rho' q}^{(0)}) \right\} \int_0^H (-\partial_z \rho') \int_0^z V(z')\mathrm{d}z'\mathrm{d}z$$
$$+2\bar{\delta}^{(0)} \left\{ \bar{\kappa}_2 + \int_0^H (-\partial_z \rho') \int_0^z B(z')\mathrm{d}z'\mathrm{d}z \right\}. \tag{4.11}$$

The first term can be associated with the movement of the free-surface displacement as in equation (4.9). The remaining terms can be interpreted as a composite effective lateral diffusivity. In particular, the stratification-induced lateral dispersion is given by the double integral

$$D^* = \int_0^H (-\partial_z \rho') \int_0^z B(z')\mathrm{d}z'\mathrm{d}z \; . \tag{4.12}$$

For stable stratification the signs of $\partial_z \rho'$ and of the incomplete integral (4.7) of $B(z')$, guarantee that D^* is non-negative. Thus, the density disturbance necessarily disperses more rapidly than marked solute. The contribution to D^* is particularly strong for density gradients $\partial_z \rho'$ close to the level at which the buoyancy-driven velocity profile $B(z)$ changes sign.

5. Estimating the stratification-induced dispersion

If the eddy viscosity ν_3 is constant, then equation (4.6) gives $\xi = 3/8$, i.e. the buoyancy-driven transverse flow $B(z)$ reaches its maximum at $z = (1 - 2\xi)H = \frac{1}{4}H$. The formula (4.12) for the stratification-induced lateral dispersion takes the neat form

$$D^* = \frac{gH^3}{48\bar{\nu}_3} \int_0^H (-\partial_z \rho') \left(\frac{z}{H}\right)^2 \left(\frac{H-z}{H}\right) \left(3 - 2\frac{z}{H}\right) \mathrm{d}z \; . \tag{5.1}$$

If the density decreases linearly from the bed to the free surface, then

$$D^* = \frac{gH^3[\rho'(0) - \rho'(H)]}{320\bar{\nu}_3} \; . \tag{5.2}$$

The inverse dependence upon the vertical mixing ν_3 of momentum is reminiscent of the inverse dependence upon the vertical mixing κ_3 of solute in the Taylor (1953) shear dispersion mechanism, for which Aris (1956) developed the method of moments.

For the Hudson estuary, figures 4 and 9 of Nepf & Geyer (1996) provide the estimates

$$H = 10\,\mathrm{m},\ \rho'(0) - \rho'(H) = 0.01,\ \bar{\nu}_3 = 0.004\,\mathrm{m}^2\mathrm{s}^{-1}\ . \tag{5.3}$$

These, together with the standard estimate $g = 10\,\mathrm{m\,s}^{-2}$, lead to the prediction

$$D^* = 78\,\mathrm{m}^2\mathrm{s}^{-1}\ . \tag{5.4}$$

Fischer (1973) shows that the lateral turbulent diffusion can be estimated as being 2.25 times the vertical average $\bar{\nu}_3$, i.e. about $0.009\,\mathrm{m}^2\ \mathrm{s}^{-1}$. Thus, the predicted stratification-induced lateral dispersion is about 8700 times the turbulent diffusivity. It is the combination of weak vertical mixing and strong stratification that makes the disparity so enormous.

A more realistic model for the turbulence is given by the von Kármán turbulence model. The eddy viscosity has a parabolic depth profile:

$$\nu_3 = 6\bar{\nu}_3(H - z)\left(\frac{z}{H} + \exp(-1/\varepsilon)\right)\ , \tag{5.5}$$

in which the roughness height is assumed to be a tiny fraction $\exp(-1/\varepsilon)$ of the total water depth H. The small parameter ε is typically about 0.11. The coefficient 6 is the result of an approximation in which constant factors formally smaller than any power of ε are ignored. To the same approximation, the coefficient ξ defined in equation (4.6) has the value

$$\xi = \frac{1}{2} - \frac{\varepsilon}{4(1-\varepsilon)}\ , \tag{5.6}$$

the buoyancy-driven velocity profile is

$$B(z) = \frac{gH^2}{24\bar{\nu}_3}\left\{1 - 2\frac{z}{H} + \frac{\varepsilon}{1-\varepsilon}\left[1 + \ln\left(\frac{z}{H} + \exp(-1/\varepsilon)\right)\right]\right\}\ , \tag{5.7}$$

and the formula (4.12) for the stratification-induced lateral dispersion becomes

$$D^* = \frac{gH^3}{24\bar{\nu}_3}\int_0^H (-\partial_z\rho')\frac{z}{H}\left\{\frac{H-z}{H} + \frac{\varepsilon}{1-\varepsilon}\ln\left(\frac{z}{H} + \exp(-1/\varepsilon)\right)\right\}\mathrm{d}z\ , \tag{5.8}$$

If the density decreases linearly from the bed to the free surface, then

$$D^* = \frac{gH^3[\rho'(0) - \rho'(H)]}{144\bar{\nu}_3}\left(1 - \frac{3\varepsilon}{2(1-\varepsilon)}\right)\ . \tag{5.9}$$

It is the low eddy viscosity near the bed that permits a faster buoyancy-driven flow and a larger numerical coefficient 1/144 in equation (5.9) than the 1/320 in (5.2).

For the von Kármán turbulence model, the estimates (5.3) from Nepf & Geyer (1996) leads to a prediction

$$D^* = 141\,\mathrm{m}^2\ \mathrm{s}^{-1}, \tag{5.10}$$

nearly 16 000 times the turbulent diffusivity.

6. Concluding remarks

When a discharge causes a density increase, there is a density-driven flow outwards near the bed and a compensating inflow near the surface. At large times the velocity profile for this lateral flow is controlled by the vertical mixing of momentum. There is an associated vertical displacement downwards near the centre and upwards at large distances which perturbs any background stratification. The resulting density perturbation spreads diffusively with a stratification-enhanced dispersion coefficient which is proportional to the stratification but inversely proportional to the vertical mixing of lateral momentum. Estimates for the Hudson estuary suggest that the effective lateral diffusivity for a density anomaly is several orders of magnitude greater than the lateral diffusivity for marked particles. This is in qualitative agreement with observations made by Geyer of salinity and temperature disturbances in the Hudson.

I am grateful to the Gratia Houghton Rinehart Coastal Research Center for financial support enabling me to visit the Woods Hole Oceanographic Institution and to Rocky Geyer for pointing out this research problem so clearly.

REFERENCES

Aris, R. 1956 On the dispersion of a solute in a fluid flowing through a tube. *Proc. R. Soc. Lond.* A **235**, 67–77.

Fischer, H. B. 1973 Longitudinal dispersion and turbulent mixing in open-channel flow. *Ann. Rev. Fluid Mech.* **5**, 59–78.

Geyer, W. R. & Nepf, H. 1996 Tidal pumping of salt in a moderately stratified estuary. *Coast. Estuarine Stud.* **53**, 213–226.

Nepf, H. & Geyer, W. R. 1996 Intratidal variations in stratification and mixing in the Hudson. *J. Geophys. Res.* **101**(C5), 12079–12086.

Smith, R. 1978*a* Effect of salt upon hot water dispersion in well-mixed estuaries. *Est. Coast Mar. Sci.* **7**, 445–454.

Smith, R. 1978*b* Effect of salt upon hot water dispersion in well-mixed estuaries. Part 2. Lateral dispersion. *Waste Heat Management and Utilization* **3**, 91–108.

Taylor, G. I. 1953 Dispersion of soluble matter in solvent flowing slowly through a tube. *Proc. R. Soc. Lond.* A **219**,186–203.